Student Name/Number: ________________________

KINROSS WOLAROI SCHOOL

2022

Year 12 Mathematics Extension 2

Trial HSC Examination

Teacher Setting Paper: Mr Bowman
Head of Department: Mr Doyle

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculator may be used
- Write your answers for Section I on the multiple-choice answer sheet provided
- A reference sheet is provided at the back of this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations.

Total marks –100

Section I – Multiple-Choice

10 marks

Attempt Questions 1-10

Allow 15 minutes for this section

Section II – Extended Response

90 marks

Attempt questions 11 – 16

Allow 2 hours and 45 minutes for this section

This examination paper does not necessarily reflect the content or format of the Higher School Certificate Examination in this subject.

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Student Name/Number:______________________________

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 - 10.

QUESTION 1

Of the following, select the statement that is **TRUE** for all real numbers x and y**:**

(A) If $x^3 > y^3$, then $x > y$

(B) If $x^2 > y^2$, then $x > y$

(C) If $x > y$, then $\frac{1}{x} < \frac{1}{y}$

(D) If $|x| > |y|$, then $x > y$

QUESTION 2

Given that $|z-2| = 2$ and $\arg(z-2) = \frac{2\pi}{3}$, which of the following is an expression for z?

(A) $z = 1 + i\sqrt{3}$

(B) $z = \left(2 + \sqrt{3}\right) + i$

(C) $z = \left(2 - \sqrt{3}\right) + i$

(D) $z = 3 + i\sqrt{3}$

QUESTION 3

$OABC$ is a rectangle with $\overrightarrow{OA} = 3\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$ and $\overrightarrow{OC} = 6\boldsymbol{i} + 4\boldsymbol{j} + a\boldsymbol{k}$ for some constant a. What is the value of a?

(A) -2

(B) -3

(C) -5

(D) -6

QUESTION 4

Which of the following is an expression for $\displaystyle\int \frac{1}{\sqrt{3-2x-x^2}}\,dx$?

(A) $\sin^{-1}\left(\frac{x-1}{2}\right)+C$

(B) $\sin^{-1}\left(\frac{x+1}{2}\right)+C$

(C) $\sin^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C$

(D) $\sin^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C$

QUESTION 5

A particle is moving in a straight line with Simple Harmonic Motion. At any time t seconds it has displacement x metres from a fixed point on the line and velocity v ms^{-1} given by $v^2 = 9 - 4(x-1)^2$. What is the amplitude of the motion?

(A) 1.5 metres

(B) 2 metres

(C) 2.5 metres

(D) 3 metres

QUESTION 6

Consider the statement

For any function f(x), f(x) is not continuous at $x = c \Rightarrow$ f(x) is not differentiable at $x = c$.

Which of the following is correct?

(A) The contrapositive statement is false and the converse statement is false.

(B) The contrapositive statement is false and the converse statement is true.

(C) The contrapositive statement is true and the converse statement is false.

(D) The contrapositive statement is true and the converse statement is true.

QUESTION 7

Which of the following is an expression for $e^{i3\theta} + e^{i\theta}$?

(A) $2sin\theta \, e^{i2\theta}$

(B) $2cos\theta \, e^{i2\theta}$

(C) $2sin2\theta \, e^{i\theta}$

(D) $2cos2\theta \, e^{i\theta}$

QUESTION 8

The points A, B, C are collinear where $\overrightarrow{OA} = \boldsymbol{i} - \boldsymbol{j}$, $\overrightarrow{OB} = -3\boldsymbol{j} - \boldsymbol{k}$ and $\overrightarrow{OC} = 2\boldsymbol{i} + a\boldsymbol{j} + b\boldsymbol{k}$ for some constants a and b. What are the values of a and b?

(A) $a = -1$ and $b = -1$

(B) $a = -1$ and $b = 1$

(C) $a = 1$ and $b = -1$

(D) $a = 1$ and $b = 1$

QUESTION 9

If $\int_1^4 f(x)dx = k$ what is the value of $\int_1^4 f(5-x)dx$?

(A) $-k$

(B) $5 - k$

(C) $k + 5$

(D) k

QUESTION 10

A stone is projected from a point O on flat ground with speed $V\sqrt{2}$ ms^{-1} at an angle $45°$ above the horizontal. The stone moves in a vertical plane under gravity where the acceleration due to gravity is g ms^{-2}. Air resistance may be neglected.

At time t seconds the position vector of the stone relative to O is $\boldsymbol{r(t)} = Vt\boldsymbol{i} + \left(Vt - \frac{1}{2}gt^2\right)\boldsymbol{j}$.

Which one of the following about the trajectory of the stone is correct?

(A) Its horizontal range is 2 times its maximum height.

(B) Its horizontal range is 3 times its maximum height.

(C) Its horizontal range is 4 times its maximum height.

(D) Its horizontal range is 5 times its maximum height.

END OF SECTION I

Student Name/Number:________________________________

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours 45 minutes for this section

Answer each question in the booklets provided. Extra writing booklets are available.

Marks

QUESTION 11 (15 marks) **Start a new writing booklet**

(a) The complex numbers z and w are represented on the Argand diagram by the points Z and W respectively. $z = 2 + 2i\sqrt{3}$ and the point W is obtained by rotating the point Z in a clockwise direction about the origin through an angle of 90°.

(i) Find z and w in modulus-argument form. **2**

(ii) Find zw and $\dfrac{z}{w}$ in modulus-argument form. **2**

(b) (i) Find $\displaystyle\int \frac{x}{\sqrt{1+3x^2}}\,dx$. **2**

(ii) **3**

Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan x\,dx$.

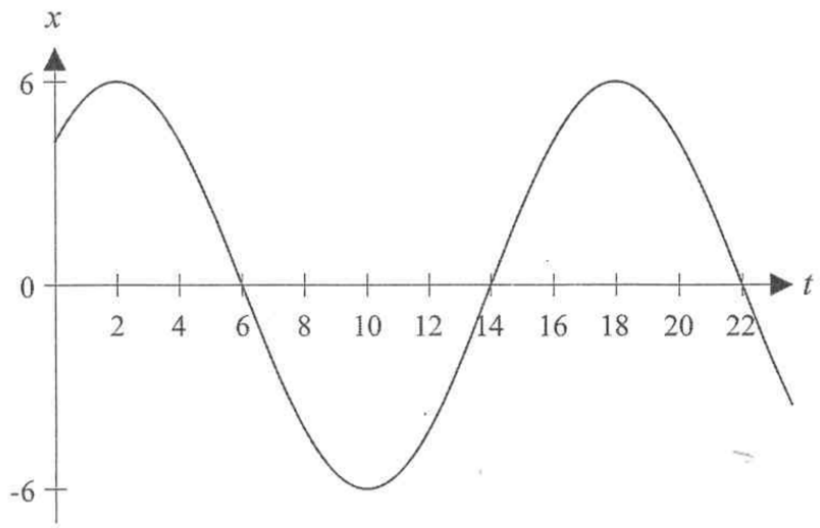
(c) Consider the equation $z^3 - 11z^2 + 55z - 125 = 0$.

(i) Find the three roots of the equation in the form $a + ib$, where a, b are real. **3**

(ii) Show that the points A, B and C in the Argand diagram representing these roots lie on a circle of the form $|z| = k$ for some constant k, and find the area of ΔABC. **3**

QUESTION 12 (15 marks) **Start a new writing booklet**

(a)



The graph shows the displacement x cm from the centre of motion at time t seconds for a particle performing Simple Harmonic Motion in a straight line.

(i) Find an expression for x as a function of t. **2**

(ii) Find the distance travelled by the particle in the first **minute** of its motion after observation began at time $t = 0$. **2**

(b)

(i) Find real numbers a, b and c such that **3**

$$\frac{1}{x^2(x-1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1}.$$

(ii) Hence find **2**

$$\int \frac{1}{x^2(x-1)}\,dx$$

(c) If $I_n = \int_1^e (1 - lnx)^n dx$ for $n = 0, 1, 2, \ldots$ **3**

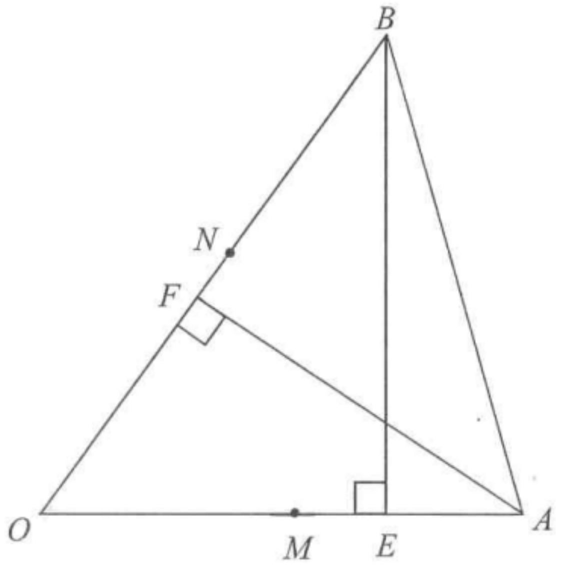
show that $I_n = -1 + nI_{n-1}$

(d) Use Mathematical Induction to prove that **3**

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\ldots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n} \text{ for } n \geq 2$$

QUESTION 13 (15 marks) **Start a new writing booklet**

(a)



Not to scale

In ΔOAB, BE is the altitude from B to OA and AF is the altitude from A to OB. **3**

M, N are the midpoints of OA, OB respectively. $\overrightarrow{OA} = \tilde{a}$ and $\overrightarrow{OB} = \tilde{b}$.
Use vector methods to show that $|OM| \times |OE| = |ON| \times |OF|$.

(b) A body of mass m kg is travelling in a horizontal straight line so that the resultant force on the body is a resistance force of magnitude $\frac{1}{10}m\sqrt{1+v}$ when its speed is v ms^{-1}. Initially the speed of the body is 15 ms^{-1}.

(i) Find the time taken for the body to come to rest. **3**

(ii) Find the distance travelled by the body in coming to rest. **3**

(c) Consider the lines L_1, L_2 determined by the vector equations

$$L_1 : \ \boldsymbol{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \text{ and } L_2 : \ \boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Show that L_1 and L_2 intersect and are perpendicular, stating the coordinates of the point of intersection. **3**

(d) (i) Find the parametric equations of the line through $(-2, 1, 2)$ that is parallel to $\boldsymbol{v} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$. **2**

(i) Determine whether either of the points $(1, 2, 3)$ or $(-6, 3, -4)$ lie on this line. **1**

QUESTION 14 (15 marks) **Start a new writing booklet**

(a) (i) Use de Moivre's Theorem to show that **3**

$$cos5\theta = 16\cos^5\theta - 20\cos^3\theta + 5cos\theta$$

(ii) Solve the equation $cos5\theta = -1$ for $0 \leq \theta \leq 2\pi$. **4**

Hence show that

$$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$$

and that

$$\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = -\frac{1}{4}.$$

(b) $P(6, 3, -4)$, $Q(3, 1, 1)$ and $R(2, -1, 3)$ are the vertices of a triangle. **4**

Find the size of the largest angle (to the nearest minute).

(c) (i) Show that **2**

$$\frac{n}{n+1}\,{}^{2n}C_{n} = {}^{2n}C_{n-1} \quad \text{for } n \geq 1$$

(ii) Hence show that **2**

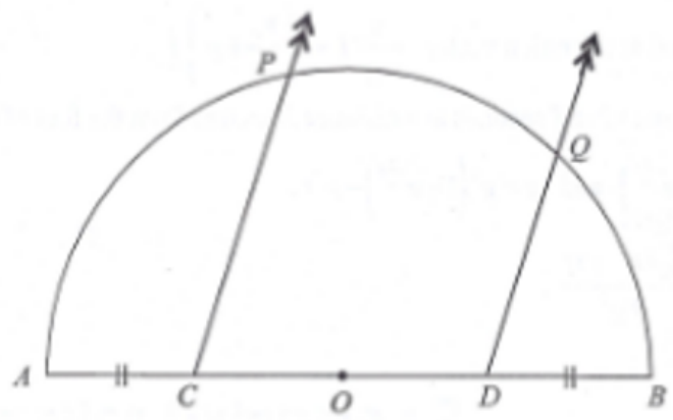
$$\frac{1}{n+1}\,{}^{2n}C_{n} \quad \text{is an integer for } n \geq 1$$

QUESTION 15 (15 marks) **Start a new writing booklet**

(a) Use the substitution $t = \tan\frac{x}{2}$ to find in simplest exact form the value of **4**

$$\int_0^{\frac{\pi}{2}} \frac{1}{3 + 5sinx}\, dx$$

(b)



A semicircle is drawn on diameter AB. O is the midpoint of AB and points C and D lie on AB such that $AC = BD$. Parallel lines are drawn through C and D intersecting the semicircle at P and Q respectively. $\overrightarrow{OC} = \tilde{c}$ and $\overrightarrow{CP} = \tilde{p}$.

(i) Explain why $\overrightarrow{DQ} = \lambda\tilde{p}$ for some scalar $\lambda > 0$ **1**

(ii) Hence show that $(1 - \lambda)\tilde{p}.\tilde{p} + 2\tilde{c}.\tilde{p} = 0$. **3**

(c) Prove by contradiction that $\log_5 7$ is irrational **3**

(d) A particle moves in a straight line. At time t seconds the particle has a displacement of x m, a velocity of v ms^{-1} and acceleration a ms^{-2}. Initially the particle has displacement 0 m and velocity 2 ms^{-1}. The acceleration is given by $a = -2e^{-x}$. The velocity of the particle is always positive.

(i) Show that $v = 2e^{-\frac{x}{2}}$. **2**

(ii) Find an expression for x as a function of t. **2**

QUESTION 16 (15 marks) **Start a new writing booklet**

(a)

(i) Show that, for all positive values of x and y, **1**

$$\frac{x+y}{2} \geq \sqrt{xy}$$

(ii) Hence prove that **2**

$$(b+c)(c+a)(a+b) \geq 8abc$$

(b) The displacement of a particle at time t is x, measured from a fixed point O where a, c are positive constants. If **5**

$$\frac{dx}{dt} = a(c^2 - x^2)$$

and $x = 0$ when $t = 0$, prove that

$$x = \frac{c(e^{2act} - 1)}{e^{2act} + 1}.$$

If $x = 3$ when $t = 1$, and $x = \frac{75}{17}$ when $t = 2$, show that $c = 5$ and evaluate a.

(c) Find an expression for **4**

$$\int \sec^3 \theta \, d\theta$$

(d) If p, q and r are positive real numbers and $p + q \geq r$, prove that **3**

$$\frac{p}{1+p} + \frac{q}{1+q} - \frac{r}{1+r} \geq 0$$

END OF PAPER.

Student Name/Number:______________________________

KINROSS WOLAROI SCHOOL

2022

Year 12 Mathematics Extension 2
Trial HSC Examination

MULTIPLE-CHOICE ANSWER SHEET

For multiple choice questions, choose the best answer A, B, C or D and fill in the correct circle.

1. A B C D
2. A B C D
3. A B C D
4. A B C D
5. A B C D
6. A B C D
7. A B C D
8. A B C D
9. A B C D
10. A B C D

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AACBA CBDDC

QUESTION 1

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(D) $z = 3 + i\sqrt{3}$

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(A) -2

(B) -3

(C) -5

(D) -6

QUESTION 4

Which of the following is an expression for $\displaystyle\int \frac{1}{\sqrt{3-2x-x^2}}\,dx$?

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QUESTION 6

Consider the statement

For any function f(x), f(x) is not continuous at $x = c \Rightarrow$ f(x) is not differentiable at $x = c$.

Which of the following is correct?

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The points A, B, C are collinear where $\overrightarrow{OA} = \boldsymbol{i} - \boldsymbol{j}$, $\overrightarrow{OB} = -3\boldsymbol{j} - \boldsymbol{k}$ and $\overrightarrow{OC} = 2\boldsymbol{i} + a\boldsymbol{j} + b\boldsymbol{k}$ for some constants a and b. What are the values of a and b?

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(B) $a = -1$ and $b = 1$

(C) $a = 1$ and $b = -1$

(D) $a = 1$ and $b = 1$

QUESTION 9

If $\int_1^4 f(x)dx = k$ what is the value of $\int_1^4 f(5-x)dx$?

(A) $-k$

(B) $5-k$

(C) $k+5$

(D) k

QUESTION 10

A stone is projected from a point O on flat ground with speed $V\sqrt{2}$ ms^{-1} at an angle $45°$ above the horizontal. The stone moves in a vertical plane under gravity where the acceleration due to gravity is g ms^{-2}. Air resistance may be neglected.

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Which one of the following about the trajectory of the stone is correct?

(A) Its horizontal range is 2 times its maximum height.

(B) Its horizontal range is 3 times its maximum height.

(C) Its horizontal range is 4 times its maximum height.

(D) Its horizontal range is 5 times its maximum height.

END OF SECTION I

MULTIPLE CHOICE DETAILED SOLUTIONS

Question	Answer	Solution	Outcomes
1	A	Function needs to be monotonic increasing or decreasing so that if $a \geq b$ then $f(a) \geq f(b)$. Only option A satisfies this requirement.	MEX12-2
2	A	$z-2=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=-1+i\sqrt{3} \qquad \therefore z=1+i\sqrt{3}$	MEX12-4
3	C	$\angle AOC=90° \quad \therefore \left(3\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}\right).\left(6\underset{\sim}{i}+4\underset{\sim}{j}+a\underset{\sim}{k}\right)=0 \quad \therefore a=-5$	MEX12-3
4	B	$\displaystyle\int\frac{1}{\sqrt{3-2x-x^2}}\,dx=\int\frac{1}{\sqrt{4-(x+1)^2}}\,dx=\sin^{-1}\left(\frac{x+1}{2}\right)+c$	MEX12-5
5	A	$v^2=9-4(x-1)^2$. $\therefore v=0$ for $x-1=\pm\frac{3}{2},\ x=-\frac{1}{2}$ or $x=\frac{5}{2}$. Particle moves 3 m between its extreme positions. Amplitude is 1.5 m.	MEX12-6
6	C	Contrapositive: *$f(x)$ is differentiable at $x=c \Rightarrow f(x)$ is continuous at $x=c$* True Converse: *$f(x)$ is not differentiable at $x=c \Rightarrow f(x)$ is not continuous at $x=c$* False	MEX12-2
7	B	$e^{i3\theta}+e^{i\theta}=\left(1+e^{i2\theta}\right)e^{i\theta}$ $\left(1+e^{i2\theta}\right)=1+\cos2\theta+i\sin2\theta=2\cos\theta\left(\cos\theta+i\sin\theta\right)=2\cos\theta\, e^{i\theta}$ $\therefore e^{i3\theta}+e^{i\theta}=2\cos\theta\, e^{i2\theta}$	MEX12-4
8	D	A, B, C collinear $\Leftrightarrow \overrightarrow{BA}=\lambda\overrightarrow{AC}$ for some scalar λ. $\overrightarrow{BA}=\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k} \qquad \overrightarrow{AC}=\underset{\sim}{i}+(a+1)\underset{\sim}{j}+b\underset{\sim}{k} \qquad \therefore a=1,\ b=1$	MEX12-3
9	D	Substituting $u=5-x$, $\displaystyle\int_1^4 f(5-x)\,dx=\int_4^1 f(u)(-1)\,du=\int_1^4 f(u)\,du=k$	MEX12-5
10	C	$\dot{y}=0 \Rightarrow V-gt=0 \quad \therefore t=\frac{V}{g}$ and $y=\frac{V^2}{g}\left(1-\frac{1}{2}\right)=\frac{V^2}{2g}$ at maximum height. $y=0 \Rightarrow Vt-\frac{1}{2}gt^2=0 \quad \therefore t=\frac{2V}{g}$ and $x=\frac{2V^2}{g}$ at horizontal range. $\frac{2V^2}{g}=4\times\frac{V^2}{2g}$. Hence horizontal range is 4 times maximum height.	MEX12-6

QUESTION 11

(a) (i)

$$z=4\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)=4\left(\cos\tfrac{\pi}{3}+i\sin\tfrac{\pi}{3}\right) \qquad w=4\left(\cos(\tfrac{\pi}{3}-\tfrac{\pi}{2})+i\sin(\tfrac{\pi}{3}-\tfrac{\pi}{2})\right)=4\left(\cos(-\tfrac{\pi}{6})+i\sin(-\tfrac{\pi}{6})\right)$$

(a) (ii)

$$zw=4\times4\left\{\cos\left(\tfrac{\pi}{3}+(-\tfrac{\pi}{6})\right)+i\sin\left(\tfrac{\pi}{3}+(-\tfrac{\pi}{6})\right)\right\}=16\left(\cos\tfrac{\pi}{6}+i\sin\tfrac{\pi}{6}\right)$$

$$\frac{z}{w}=\frac{4}{4}\left\{\cos\left(\tfrac{\pi}{3}-(-\tfrac{\pi}{6})\right)+i\sin\left(\tfrac{\pi}{3}-(-\tfrac{\pi}{6})\right)\right\}=\cos\tfrac{\pi}{2}+i\sin\tfrac{\pi}{2}$$

(b) **(i)**

$$\int \frac{x}{\sqrt{1+3x^2}}\,dx$$
$$=\frac{1}{6}\int 6x(1+3x^2)^{-\frac{1}{2}}\,dx$$
$$=\frac{1}{6}(1+3x^2)^{\frac{1}{2}}\times 2+C$$
$$=\frac{1}{3}\sqrt{1+3x^2}+C$$

(b)(ii)

$$\int_0^{\frac{\pi}{4}} tanx\,dx$$
$$=[\ln cosx]_{\frac{\pi}{4}}^{0}$$
$$=ln1-\ln\left(\frac{1}{\sqrt{2}}\right)$$
$$=\frac{1}{2}ln2$$

(c) (i)

$$z^3-11z^2+55z-125=0$$
$$P(5)=0\ hence\ (z-5)\ is\ a\ factor$$

$$z^3-11z^2+55z-125=(z-5)(z^2+az+25)$$

Equating terms in z^2:

$$-11=a-5$$

Hence

$$a=-6$$

$$(z-5)(z^2-6z+25)=0$$

$$z=5, z=3\pm4i$$

QUESTION 12

(a)(i)

Amplitude is 6 cm. Period is 16 s. $\therefore \frac{2\pi}{n}=16 \quad \therefore n=\frac{\pi}{8}$.

Graph is a translation 2 units to the right of $x=6\cos\frac{\pi}{8}t$. $\quad \therefore x=6\cos\frac{\pi}{8}(t-2)$

(a)(ii)

$60=16\times3+8+4$ Hence 60 seconds is the time for 3 oscillations, plus one half an oscillation, plus twice the time taken to travel from its initial position to its position at time $t=2$.

$t=0 \Rightarrow x=6\cos\left(-\frac{\pi}{4}\right)=3\sqrt{2}$. Hence particle travels $\left(6-3\sqrt{2}\right)$ cm in first 2 seconds.

The distance travelled in $3\frac{1}{2}$ oscillations is $3\frac{1}{2}\times24\text{ cm}=84\text{ cm}$.

Hence distance travelled in the first minute is $84+2\left(6-3\sqrt{2}\right)\text{cm}=\left(96-6\sqrt{2}\right)\text{cm}$

(b)(i)

$$\frac{1}{x^2(x-1)}=\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x-1}.$$
$$1=ax(x-1)+b(x-1)+cx^2$$

Subst $x=1$

$$\boldsymbol{1=c}$$

Equate coeffs of x^2

$$a+c=0$$
$$\boldsymbol{a=-1}$$

Equate constants:

$$1=-b$$
$$\boldsymbol{b=-1}$$
$$\frac{1}{x^2(x-1)}=\frac{-1}{x}-\frac{1}{x^2}+\frac{1}{x-1}$$

(b)(ii)

$$\int\frac{1}{x^2(x-1)}dx$$
$$=\int-\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x-1}\,dx$$
$$=-lnx+\frac{1}{x}+\ln(x-1)+C$$
$$=\frac{1}{x}+\ln\left(\frac{x-1}{x}\right)+C$$

(c)

$$I_n = \int_1^e (1-\ln x)^n\,dx$$

$$= \left[x(1-\ln x)^n\right]_1^e - \int_1^e nx(1-\ln x)^{n-1}\left(-\frac{1}{x}\right)dx\ , \quad n=1,2,3,\ldots$$

$$= -1 + n\int_1^e (1-\ln x)^{n-1}\,dx$$

$$= -1 + nI_{n-1}$$

(d) To prove that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n} \text{ for } n \geq 2$$

$$n = 2$$

$$LHS = 1 - \frac{1}{2^2} = \frac{3}{4}$$

$$RHS = \frac{3}{4}$$

Hence true for $n = 2$

Assume true for $n = k$, i.e. assume that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k} \ldots\ldots\ldots.(A)$$

To prove true for $n = k+1$ i.e. that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2(k+1)} \ldots\ldots\ldots.(B)$$

LHS of equation (B)

$$= \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$= \left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right) \text{ using assumption } (A)$$

$$= \left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$= \left(\frac{k+1}{2k}\right)\left(\frac{k^2+2k}{(k+1)^2}\right)$$

$$= \left(\frac{k+1}{2k}\right)\left(\frac{k(k+2)}{(k+1)^2}\right)$$

$$= \frac{k+2}{2(k+1)} \text{ as required}$$

Hence if true for $n = k$, then true for $n = k+1$.
True for $n = 2$, therefore true for all integral $n \geq 2$.

QUESTION 13

(a)

$\overrightarrow{OM} = \frac{1}{2}\underset{\sim}{a}$ and $\overrightarrow{OE} = \lambda\underset{\sim}{a}$ for some scalar λ . Then $\left|OM\right|\left|OE\right| = \frac{1}{2}\lambda\, \underset{\sim}{a} \cdot \underset{\sim}{a}$

$\overrightarrow{ON} = \frac{1}{2}\underset{\sim}{b}$ and $\overrightarrow{OF} = \mu\underset{\sim}{b}$ for some scalar μ . Then $\left|ON\right|\left|OF\right| = \frac{1}{2}\mu\, \underset{\sim}{b} \cdot \underset{\sim}{b}$

$\overrightarrow{AF} \perp \overrightarrow{OB} \quad \therefore \left(\mu\, \underset{\sim}{b} - \underset{\sim}{a}\right) \cdot \underset{\sim}{b} = 0 \quad \therefore \mu\, \underset{\sim}{b} \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot \underset{\sim}{b}$

Similarly $\overrightarrow{BE} \perp \overrightarrow{OA}$ gives $\lambda\, \underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{a}$. But $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a}$. Hence $\left|OM\right|\left|OE\right| = \left|ON\right|\left|OF\right| = \frac{1}{2}\underset{\sim}{a} \cdot \underset{\sim}{b}$

(b)(i)

Let the body come to rest in T seconds

$$\ddot{x} = -\tfrac{1}{10}\sqrt{1+v}$$

$$\frac{dv}{dt} = -\tfrac{1}{10}\sqrt{1+v}$$

$$\int_{15}^{0} \frac{1}{\sqrt{1+v}}\, dv = -\tfrac{1}{10}\int_0^T dt$$

$$2\left[\sqrt{1+v}\right]_{15}^{0} = -\tfrac{1}{10}T$$

$$2(1-4) = -\tfrac{1}{10}T$$

$$T = 60$$

Hence body comes to rest in 1 minute.

(b)(ii)

Let the body come to rest after travelling X metres.

$$v\frac{dv}{dx} = -\tfrac{1}{10}\sqrt{1+v}$$

$$\int_{15}^{0} \frac{v}{\sqrt{1+v}}\, dv = -\tfrac{1}{10}\int_0^X dx$$

$$\int_{15}^{0}\left(\sqrt{1+v} - \frac{1}{\sqrt{1+v}}\right) dv = -\tfrac{1}{10}X$$

$$\left[\tfrac{2}{3}(1+v)^{\frac{3}{2}} - 2\sqrt{1+v}\right]_{15}^{0} = -\tfrac{1}{10}X$$

$$\tfrac{2}{3}(1-64) - 2(1-4) = -\tfrac{1}{10}X$$

$$X = 360$$

Hence body travels 360 m in coming to rest.

(c)

At any intersection point
$$\begin{aligned} 3+2\lambda &= -1+\mu \quad (1) \\ 2-\lambda &= 1+\mu \quad (2) \\ -1+\lambda &= -\mu \quad (3) \end{aligned}$$
is a set of 3 consistent simultaneous equations.

Considering (1) and (2) : (1) – (2) $\Rightarrow 1+3\lambda=-2$ and hence $\lambda=-1,\ \mu=2$.
Substituting in (3) : LHS = –2 = RHS

Hence the set of 3 equations is consistent with solution $\lambda=-1,\ \mu=2$.

At intersection $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$. Also $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2-1-1=0$.

Hence the lines intersect at right angles at $(1,3,-2)$

(d)(i)

$$\frac{x+2}{2}=\frac{y-1}{-1}=\frac{z-2}{3}=\lambda$$

Hence

$$x=2\lambda-2, y=1-\lambda, z=3\lambda+2$$

(d)(ii)

Substituting $(1,2,3)$ gives

$$1=2\lambda-2, 2=1-\lambda, 3=3\lambda+2$$
$$\lambda=\frac{3}{2},-1,\frac{1}{3}$$

$\lambda\ is\ not\ the\ same$ hence point $(1,2,3)$ does **NOT** lie on line

Substituting $(-6,3,-4)$ gives

$$-6=2\lambda-2, 3=1-\lambda, -4=3\lambda+2$$
$$\lambda=-2,-2,-2$$

$\lambda\ is\ the\ same$ hence point $(-6,3,-4)$ **DOES** lie on line.

QUESTION 14

(a)(i)

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

$$\begin{aligned}
\therefore \cos 5\theta &= \text{Re}(\cos\theta + i\sin\theta)^5 \\
&= {}^5C_0\cos^5\theta + {}^5C_2\cos^3\theta(i\sin\theta)^2 + {}^5C_4\cos\theta(i\sin\theta)^4 \\
&= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 \\
&= \cos^5\theta(1+10+5) + \cos^3\theta(-10-10) + 5\cos\theta \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta
\end{aligned}$$

(a)(ii)

$\cos 5\theta = -1$ for $5\theta = n\pi,\ n = 1, 3, 5, \ldots$ $\quad \therefore \theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$ for $0 \le \theta \le 2\pi$.

Hence $16x^5 - 20x^3 + 5x + 1 = 0$ has roots $\cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, -1, \cos\frac{7\pi}{5}, \cos\frac{9\pi}{5}$.

But $\cos\frac{7\pi}{5} = \cos\left(2\pi - \frac{3\pi}{5}\right) = \cos\frac{3\pi}{5}$ and similarly $\cos\frac{9\pi}{5} = \cos\frac{\pi}{5}$

Hence using the relationships between coefficients and roots :

$$2\cos\tfrac{\pi}{5} + 2\cos\tfrac{3\pi}{5} - 1 = 0$$

$$\therefore \quad \cos\tfrac{\pi}{5} + \cos\tfrac{3\pi}{5} = \tfrac{1}{2}$$

$$-\left(\cos\tfrac{\pi}{5}\cos\tfrac{3\pi}{5}\right)^2 = -\tfrac{1}{16}$$

$$\therefore \quad \cos\tfrac{\pi}{5}\cos\tfrac{3\pi}{5} = \pm\tfrac{1}{4}$$

But $\cos\frac{\pi}{5} > 0$ and $\cos\frac{3\pi}{5} < 0$

$$\therefore \quad \cos\tfrac{\pi}{5}\cos\tfrac{3\pi}{5} = -\tfrac{1}{4}$$

(b)

$P(6, 3, -4)$, $Q(3, 1, 1)$ and $R(2, -1, 3)$

Largest angle is opposite largest side.

Side lengths:

$$PQ^2 = 38$$
$$QR^2 = 9$$
$$PR^2 = 81$$

PR longest side hence angle PQR is required angle.

$$\overrightarrow{PQ} = -3\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k} \ \boldsymbol{so}\ \boldsymbol{PQ} = \sqrt{\mathbf{38}}$$
$$\overrightarrow{QR} = -\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k} \ \boldsymbol{so}\ \boldsymbol{QR} = \mathbf{3}$$

$$cos\angle PQR = \frac{3^2 + 38 - 9^2}{2 \times 3 \times \sqrt{38}}$$
$$= -\frac{17}{3\sqrt{38}}$$
$$\theta = 156°49'$$

(c)(i)

$$\frac{n}{n+1}\,{}^{2n}C_n = \frac{n(2n)!}{(n+1)n!\,n!} = \frac{(2n)!}{(n+1)!(n-1)!} = {}^{2n}C_{n-1}$$

(c)(ii)

$$\frac{1}{n+1}\,{}^{2n}C_n = \left(1-\frac{n}{n+1}\right){}^{2n}C_n = {}^{2n}C_n - {}^{2n}C_{n-1}$$

Since these binomial coefficients are respectively the number of ways of choosing n or $(n-1)$ items from $2n$ items, both are integers hence their difference is also an integer.

QUESTION 15

(a)

$$t=\tan\tfrac{x}{2}$$
$$dt=\tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx$$
$$\frac{2}{1+t^2}dt=dx$$

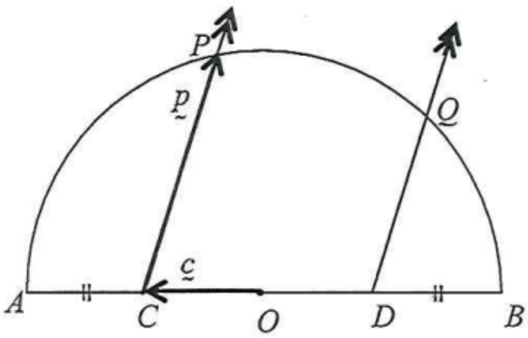
$$x=0\Rightarrow t=0$$
$$x=\tfrac{\pi}{2}\Rightarrow t=1$$

$$3+5\sin x=\frac{3\left(1+t^2\right)+10t}{1+t^2}$$
$$=\frac{(3t+1)(t+3)}{1+t^2}$$
$$\frac{1}{3+5\sin x}=\left(\frac{1+t^2}{2}\right)\left\{\frac{3}{3t+1}-\frac{1}{t+1}\right\}$$

$$\int_0^{\frac{\pi}{2}}\frac{1}{3+5\sin x}\,dx=\int_0^1\left\{\frac{3}{3t+1}-\frac{1}{t+1}\right\}dt$$
$$=\left[\ln\frac{3t+1}{t+1}\right]_0^1$$
$$=\ln 2-\ln 1$$
$$=\ln 2$$

(b)(i) & (ii)

Answer



$\overrightarrow{DQ}$, $\overrightarrow{CP}$ are parallel in the same direction.

Hence $\overrightarrow{DQ}=\lambda\underset{\sim}{p}$ for some $\lambda>0$.

Then $\overrightarrow{OP}=\underset{\sim}{c}+\underset{\sim}{p}$ and $\overrightarrow{OQ}=-\underset{\sim}{c}+\lambda\underset{\sim}{p}$

But $\left|\overrightarrow{OP}\right|=\left|\overrightarrow{OQ}\right|$ (radii of the circle)

$$\therefore\quad\left(\underset{\sim}{c}+\underset{\sim}{p}\right).\left(\underset{\sim}{c}+\underset{\sim}{p}\right)=\left(-\underset{\sim}{c}+\lambda\underset{\sim}{p}\right).\left(-\underset{\sim}{c}+\lambda\underset{\sim}{p}\right)$$
$$\underset{\sim}{c}.\underset{\sim}{c}+\underset{\sim}{p}.\underset{\sim}{p}+2\underset{\sim}{c}.\underset{\sim}{p}=\underset{\sim}{c}.\underset{\sim}{c}+\lambda^2\underset{\sim}{p}.\underset{\sim}{p}-2\lambda\,\underset{\sim}{c}.\underset{\sim}{p}$$
$$\left(1-\lambda^2\right)\underset{\sim}{p}.\underset{\sim}{p}+2(1+\lambda)\underset{\sim}{c}.\underset{\sim}{p}=0$$
$$(1-\lambda)\underset{\sim}{p}.\underset{\sim}{p}+2\underset{\sim}{c}.\underset{\sim}{p}=0$$

(c.) Let $\log_5 7=p$ and assume this is rational, i.e. of form $\frac{x}{y}$.

$$7=5^{\frac{x}{y}}$$
$$7^y=5^x$$

Which is impossible as all powers of 7 end in a 1, 3, 7 or 9 whereas all powers of 5 end in a 5.
Hence $log_5 7\ is\ irrational$

(d)(i)

$$a=-2e^{-x}$$
$$v.\frac{dv}{dx}=-2e^{-x}$$
$$vdv=-2e^{-x}dx$$
$$\int_2^v vdv=-\int_0^x 2e^{-x}dx$$
$$\frac{v^2}{2}-2=2e^{-x}-2$$
$$v^2=4e^{-x}$$
$$v=2e^{-\frac{x}{2}}$$

(d)(ii)

$$v = 2e^{-\frac{x}{2}}$$

$$\frac{dx}{dt} = 2e^{-\frac{x}{2}}$$

$$\frac{1}{2}e^{\frac{x}{2}}dx = dt$$

$$\int_0^x \frac{1}{2}e^{\frac{x}{2}}dx = \int_0^t dt$$

$$e^{\frac{x}{2}} - 1 = t$$

$$e^{\frac{x}{2}} = t + 1$$

$$\frac{x}{2} = \ln(t+1)$$

$$x = 2\ln(t+1)$$

QUESTION 16

(a)(i)

$$since\ (x-y)^2 \geq 0$$
$$\therefore x^2 + y^2 \geq 2xy$$
$$x^2 + 2xy + y^2 \geq 4xy$$
$$(x+y)^2 \geq 4xy$$
$$(x+y) \geq 2\sqrt{xy}$$
$$\frac{x+y}{2} \geq \sqrt{xy}\ \ as\ required.$$

(a) (ii)

To prove $(b+c)(c+a)(a+b) \geq 8abc$

From (a)(i),

$$\frac{b+c}{2} \geq \sqrt{bc}$$
$$i.e. b+c \geq 2\sqrt{bc}$$

Similarly,

$$c+a \geq 2\sqrt{ca}$$
$$a+b \geq 2\sqrt{ab}$$

Hence

$$(b+c)(c+a)(a+b) \geq 8\sqrt{a^2b^2c^2}$$
$$i.e.\ (b+c)(c+a)(a+b) \geq 8abc\ \ as\ required.$$

(b)

$$\frac{dx}{dt} = a(c^2 - x^2)$$

$$\frac{dt}{dx} = \frac{1}{a}.\frac{1}{c^2 - x^2}$$

Integrating,

$$t = \frac{1}{2ac}\ln\left(\frac{c+x}{c-x}\right) + C$$
$$t = 0\ when\ x = 0$$
$$c = 0$$
$$t = \frac{1}{2ac}\ln\left(\frac{c+x}{c-x}\right)$$
$$\frac{c+x}{c-x} = e^{2act}$$
$$c + x = (c-x)e^{2act}$$
$$c(2e^{act} - 1) = x(e^{2act} + 1)$$
$$x = \frac{c(e^{2act} - 1)}{e^{2act} + 1}\ \ as\ required.$$

$$x = 3\ when\ t = 1$$

$$3 = \frac{c(e^{2ac} - 1)}{e^{2ac} + 1} \ldots .(i)$$

$$x = \frac{75}{17} \; when \; t = 2$$

$$\frac{75}{17} = \frac{c(e^{4ac} - 1)}{e^{4ac} + 1} \ldots . (ii)$$

From (i),

$$ce^{2ac} - c = 3e^{2ac} + 3$$
$$e^{2ac} = \frac{c+3}{c-3} \ldots . (iii)$$

From (ii)

$$75e^{4ac} + 75 = 17ce^{4ac} - 17c$$
$$e^{4ac} = \frac{17c + 75}{17c - 75} \ldots \; (iv)$$

Since

$$e^{4ac} = (e^{2ac})^2 \text{ from (iii)and (iv) we have}$$

$$\frac{17c + 75}{17c - 75} = \frac{(c+3)^2}{(c-3)^2}$$
$$(c^2 - 6c + 9)(17c + 75) = (c^2 + 6c + 9)(17c - 75)$$
$$17c^3 + 75c^2 - 102c^2 - 450c + 153c + 675 = 17c^3 - 75c^2 + 102c^2 - 450c + 153c - 675$$
$$54c^2 - 1350 = 0$$
$$c = 5$$

$$e^{10a} = 4$$
$$10a = ln4$$
$$a = \frac{1}{5} ln2$$

(c)

$$Let \; I = \int sec^3 x \, dx$$
$$= \int \quad secx. \sec^2 x \, dx$$
$$= secxtanx - \int secxtanx. tanxdx$$
$$= secxtanx - \int secx(\sec^2 x - 1)dx$$
$$= secxtanx - \int \sec^3 xdx + \int secx \, dx$$
$$I = secxtanx - I + \ln(secx + tanx) + C$$
$$2I = secxtanx + \ln(secx + tanx) + C$$
$$\int \sec^3 xdx = \frac{1}{2}(secxtanx + \ln[secx + tanx]) + C$$

(d)

$$To\ prove\ \frac{p}{1+p}+\frac{q}{1+q}-\frac{r}{1+r}\geq 0\ given\ p+q\geq r$$

(This is one of many ways)

$$\begin{aligned}
\frac{p}{1+p}+\frac{q}{1+q}-\frac{r}{1+r} &= \frac{p(1+q)(1+r)+q(1+p)(1+r)-r(1+p)(1+q)}{(1+p)(1+q)(1+r)}\\
&= \frac{p+pr+pq+pqr+q+qr+pq+pqr-r-rq-rp-pqr}{(1+p)(1+q)(1+r)}\\
&= \frac{p+2pq+q-r}{(1+p)(1+q)(1+r)}\\
&\geq \frac{r+2pq-r}{(1+p)(1+q)(1+r)}\quad since\ p+q\geq r\\
&= \frac{2pq}{(1+p)(1+q)(1+r)}\\
&\geq 0\ since\ p,q,r>0\ (given)
\end{aligned}$$

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours 45 minutes for this section

Answer each question in the booklets provided. Extra writing booklets are available.

Marks

QUESTION 11 (15 marks) **Start a new writing booklet**

(a) The complex numbers z and w are represented on the Argand diagram by the points Z and W respectively. $z = 2 + 2i\sqrt{3}$ and the point W is obtained by rotating the point Z in a clockwise direction about the origin through an angle of 90°.

(i) Find z and w in modulus-argument form. **2**

(ii) Find zw and $\dfrac{z}{w}$ in modulus-argument form. **2**

(b) (i) Find $\displaystyle\int \frac{x}{\sqrt{1+3x^2}}\,dx$. **2**

(ii) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \tan x \, dx$. **3**

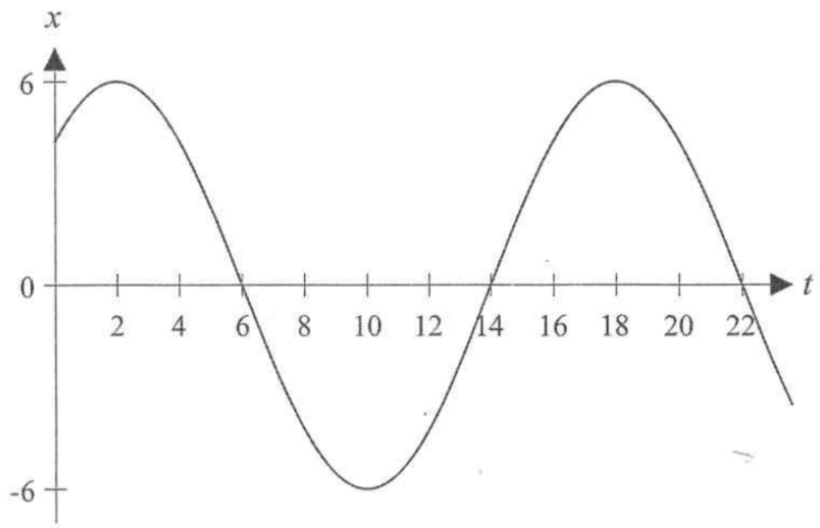
(c) Consider the equation $z^3 - 11z^2 + 55z - 125 = 0$.

(i) Find the three roots of the equation in the form $a + ib$, where a, b are real. **3**

(ii) Show that the points A, B and C in the Argand diagram representing these roots lie on a circle of the form $|z| = k$ for some constant k, and find the area of ΔABC. **3**

QUESTION 12 (15 marks) **Start a new writing booklet**

(a)



The graph shows the displacement x cm from the centre of motion at time t seconds for a particle performing Simple Harmonic Motion in a straight line.

(i) Find an expression for x as a function of t. **2**

(ii) Find the distance travelled by the particle in the first **minute** of its motion after observation began at time $t = 0$. **2**

(b)

(i) Find real numbers a, b and c such that **3**

$$\frac{1}{x^2(x-1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1}.$$

(ii) Hence find **2**

$$\int \frac{1}{x^2(x-1)}\, dx$$

(c) If $I_n = \int_1^e (1 - lnx)^n dx$ for $n = 0, 1, 2, \ldots$ **3**

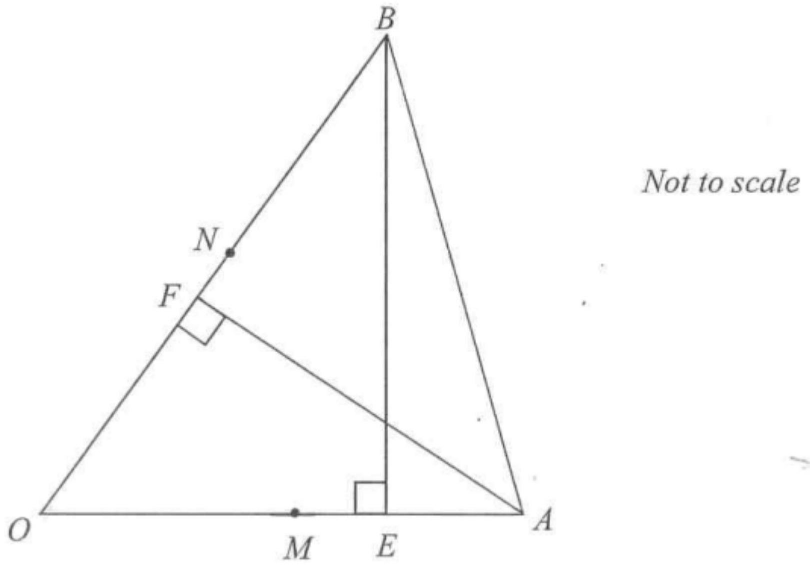
show that $I_n = -1 + nI_{n-1}$

(d) Use Mathematical Induction to prove that **3**

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \ldots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n} \text{ for } n \geq 2$$

QUESTION 13 (15 marks) **Start a new writing booklet**

(a)



In ΔOAB, BE is the altitude from B to OA and AF is the altitude from A to OB. **3**

M, N are the midpoints of OA, OB respectively. $\overrightarrow{OA} = \tilde{a}$ and $\overrightarrow{OB} = \tilde{b}$.
Use vector methods to show that $|OM| \times |OE| = |ON| \times |OF|$.

(b) A body of mass m kg is travelling in a horizontal straight line so that the resultant force on the body is a resistance force of magnitude $\frac{1}{10}m\sqrt{1+v}$ when its speed is v ms^{-1}. Initially the speed of the body is 15 ms^{-1}.

(i) Find the time taken for the body to come to rest. **3**

(ii) Find the distance travelled by the body in coming to rest. **3**

(c) Consider the lines L_1, L_2 determined by the vector equations

$$L_1 : \ \boldsymbol{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \text{ and } L_2 : \ \boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Show that L_1 and L_2 intersect and are perpendicular, stating the coordinates of the point of intersection. **3**

(d) (i) Find the parametric equations of the line through $(-2, 1, 2)$ that is parallel to $\boldsymbol{v} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$. **2**

(i) Determine whether either of the points $(1, 2, 3)$ or $(-6, 3, -4)$ lie on this line. **1**

QUESTION 14 (15 marks) **Start a new writing booklet**

(a) (i) Use de Moivre's Theorem to show that **3**

$$cos5\theta = 16\cos^5\theta - 20\cos^3\theta + 5cos\theta$$

(ii) Solve the equation $cos5\theta = -1$ for $0 \leq \theta \leq 2\pi$. **4**

Hence show that

$$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$$

and that

$$\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = -\frac{1}{4}.$$

(b) $P(6,3,-4)$, $Q(3,1,1)$ and $R(2,-1,3)$ are the vertices of a triangle. **4**

Find the size of the largest angle (to the nearest minute).

(c) (i) Show that **2**

$$\frac{n}{n+1}\,{}^{2n}C_{n} = {}^{2n}C_{n-1} \quad \text{for } n \geq 1$$

(ii) Hence show that **2**

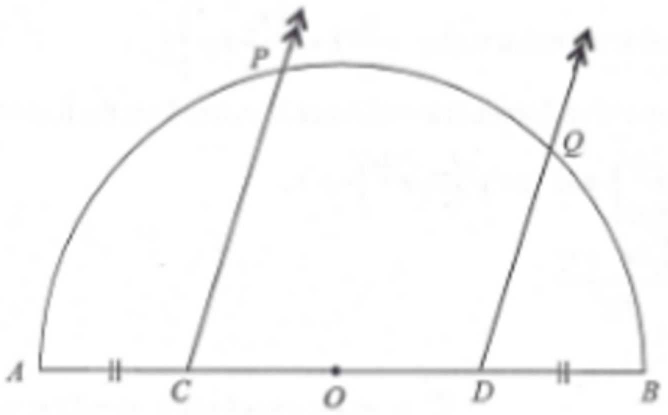
$$\frac{1}{n+1}\,{}^{2n}C_{n} \quad \text{is an integer for } n \geq 1$$

QUESTION 15 (15 marks) **Start a new writing booklet**

(a) Use the substitution $t = \tan\frac{x}{2}$ to find in simplest exact form the value of **4**

$$\int_0^{\frac{\pi}{2}} \frac{1}{3 + 5sinx}\, dx$$

(b)



A semicircle is drawn on diameter AB. O is the midpoint of AB and points C and D lie on AB such that $AC = BD$. Parallel lines are drawn through C and D intersecting the semicircle at P and Q respectively. $\overrightarrow{OC} = \tilde{c}$ and $\overrightarrow{CP} = \tilde{p}$.

(i) Explain why $\overrightarrow{DQ} = \lambda\tilde{p}$ for some scalar $\lambda > 0$ **1**

(ii) Hence show that $(1 - \lambda)\tilde{p}.\tilde{p} + 2\tilde{c}.\tilde{p} = 0$. **3**

(c) Prove by contradiction that $\log_5 7$ is irrational **3**

(d) A particle moves in a straight line. At time t seconds the particle has a displacement of x m, a velocity of v ms^{-1} and acceleration a ms^{-2}. Initially the particle has displacement 0 m and velocity 2 ms^{-1}. The acceleration is given by $a = -2e^{-x}$. The velocity of the particle is always positive.

(i) Show that $v = 2e^{-\frac{x}{2}}$. **2**

(ii) Find an expression for x as a function of t. **2**

QUESTION 16 (15 marks) **Start a new writing booklet**

(a)

(i) Show that, for all positive values of x and y, **1**

$$\frac{x+y}{2} \geq \sqrt{xy}$$

(ii) Hence prove that **2**

$$(b+c)(c+a)(a+b) \geq 8abc$$

(b) The displacement of a particle at time t is x, measured from a fixed point O where a, c are positive constants. If **5**

$$\frac{dx}{dt} = a(c^2 - x^2)$$

and $x = 0$ when $t = 0$, prove that

$$x = \frac{c(e^{2act} - 1)}{e^{2act} + 1}.$$

If $x = 3$ when $t = 1$, and $x = \frac{75}{17}$ when $t = 2$, show that $c = 5$ and evaluate a.

(c) Find an expression for **4**

$$\int \sec^3 \theta \, d\theta$$

(d) If p, q and r are positive real numbers and $p + q \geq r$, prove that **3**

$$\frac{p}{1+p} + \frac{q}{1+q} - \frac{r}{1+r} \geq 0$$

END OF PAPER.

KINROSS WOLAROI SCHOOL

2022

Year 12 Mathematics Extension 2

Trial HSC Examination

MULTIPLE-CHOICE ANSWER SHEET

For multiple choice questions, choose the best answer A, B, C or D and fill in the correct circle.

1. Ⓐ Ⓑ Ⓒ Ⓓ
2. Ⓐ Ⓑ Ⓒ Ⓓ
3. Ⓐ Ⓑ Ⓒ Ⓓ
4. Ⓐ Ⓑ Ⓒ Ⓓ
5. Ⓐ Ⓑ Ⓒ Ⓓ
6. Ⓐ Ⓑ Ⓒ Ⓓ
7. Ⓐ Ⓑ Ⓒ Ⓓ
8. Ⓐ Ⓑ Ⓒ Ⓓ
9. Ⓐ Ⓑ Ⓒ Ⓓ
10. Ⓐ Ⓑ Ⓒ Ⓓ